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# Inequalities for a polynomial and its derivative

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**Abstract**

For a polynomial $p(z)$ of degree $n$ which has no zeros in $|z| < 1$, Liman *et al.* (Appl. Math. Comput. 218:949-955, 2011) established

$$\begin{aligned}
&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}p(rz)\right| \\
&\quad\le \frac{1}{2}\left\{\left[\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\right|\right.\right. \\
&\qquad + \left.\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}\right|\right]\max_{|z|=1}|p(z)| - \left[\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\right|\right. \\
&\qquad - \left.\left.\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}\right|\right]\min_{|z|=1}|p(z)|\right\},
\end{aligned}$$

for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $R > r \ge 1$ and $|z| = 1$. In this paper, we extend the above inequality for the polynomials having no zeros in $|z| < k$, $k \le 1$. Our result generalizes certain well-known polynomial inequalities.

**MSC:** 30A10; 30C10; 30D15

**Keywords:** polynomial; inequality; maximum modulus; derivative; restricted zeros

## 1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree $n$ and $p'(z)$ be its derivative. Then it is well known that

$$\max_{|z|=1}|p'(z)| \le n \max_{|z|=1}|p(z)|, \tag{1.1}$$

and

$$\max_{|z|=R>1}|p(z)| \le R^n \max_{|z|=1}|p(z)|. \tag{1.2}$$

Inequality (1.1) is a famous result due to Bernstein [1], whereas inequality (1.2) is a simple consequence of the maximum modulus principle (see [2]). Both the above inequalities are sharp, and an equality in each holds for the polynomials having all their zeros at the origin.

For the class of polynomials having no zeros in $|z| < 1$, inequalities (1.1) and (1.2) have respectively been replaced by

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2}\max_{|z|=1}|p(z)|, \tag{1.3}$$

and

$$\max_{|z|=R>1}\left|p(z)\right| \le \frac{R^n+1}{2}\max_{|z|=1}\left|p(z)\right|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [3], whereas inequality (1.4) was proved by Ankeny and Rivlin [4], for which they made use of (1.3). Both these inequalities are also sharp, and an equality in each holds for polynomials having all their zeros on $|z| = 1$.

As an extension to (1.3) and (1.4), Malik [5] and Shah [6], respectively, proved that if $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{1+k}\max_{|z|=1}\left|p(z)\right|, \tag{1.5}$$

and

$$\max_{|z|=R>1}\left|p(z)\right| \le \frac{R^n+k}{1+k}\max_{|z|=1}\left|p(z)\right|. \tag{1.6}$$

Aziz and Dawood [7] refined inequalities (1.3) and (1.4) by proving that if $p(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{2}\left\{\max_{|z|=1}\left|p(z)\right| - \min_{|z|=1}\left|p(z)\right|\right\}, \tag{1.7}$$

and

$$\max_{|z|=R>1}\left|p(z)\right| \le \left(\frac{R^n+1}{2}\right)\max_{|z|=1}\left|p(z)\right| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}\left|p(z)\right|. \tag{1.8}$$

Both these inequalities are also sharp, and an equality in each holds for $p(z) = \alpha z^n + \gamma$ with $|\alpha| = |\gamma|$.

As a refinement of inequalities (1.7) and (1.8), Dewan and Hans [8, 9] proved that under the same assumptions, for every $|\beta| \leq 1$, $R > 1$ and $|z| = 1$, we have

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \\ &\quad\le \frac{n}{2}\left\{\left(\left|\frac{\beta}{2}\right| + \left|1+\frac{\beta}{2}\right|\right)\max_{|z|=1}\left|p(z)\right| - \left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)\min_{|z|=1}\left|p(z)\right|\right\},\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}&\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| \\ &\quad\le \frac{1}{2}\left\{\left(\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right)\max_{|z|=1}\left|p(z)\right|\right. \\ &\qquad\left. - \left(\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right)\min_{|z|=1}\left|p(z)\right|\right\}.\end{aligned} \tag{1.10}$$

Both these inequalities are also sharp, and an equality in each holds for polynomials having all their zeros on $|z| = 1$.

Liman *et al.* [10] further generalized inequalities (1.9) and (1.10) by proving that if $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| = 1$, we have

$$\begin{aligned}
&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}p(rz)\right| \\
&\quad\leq \frac{1}{2}\left\{\left[\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\right|\right.\right. \\
&\qquad + \left.\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}\right|\right]\max_{|z|=1}|p(z)| \\
&\qquad - \left[\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\right|\right. \\
&\qquad - \left.\left.\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}\right|\right]\min_{|z|=1}|p(z)|\right\}.
\end{aligned} \tag{1.11}$$

As an extension to inequality (1.11), we propose the following result.

**Theorem 1** *If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$, we have*

$$\begin{aligned}
&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\
&\quad\leq \frac{1}{2}\left\{\left[k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\right.\right. \\
&\qquad + \left.\left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right]\max_{|z|=1}|p(z)| \\
&\qquad - \left[k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\right. \\
&\qquad - \left.\left.\left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right]\min_{|z|=k}|p(z)|\right\}.
\end{aligned} \tag{1.12}$$

If we take $k = 1$ in Theorem 1, then inequality (1.12) reduces to (1.11).

Theorem 1 reduces to the following result by taking $\alpha = 1$.

**Corollary 1.1** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$, we have

$$\begin{aligned}
&\left|p(Rz) - p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - 1\right\}p(rz)\right| \\
&\quad\leq \frac{1}{2}\left\{\left[k^{-n}\left|R^n - r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - 1\right\}r^n\right|\right.\right. \\
&\qquad + \left.\left|\beta\left\{\left(\frac{R+k}{r+k}\right)^n - 1\right\}\right|\right]\max_{|z|=1}|p(z)|
\end{aligned}$$

$$- \left[ k^{-n} \left| R^n - r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - 1 \right\} r^n \right| - \left| \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - 1 \right\} \right| \right] \min_{|z|=k} \left| p(z) \right| \Bigg\}. \tag{1.13}$$

Dividing both sides of inequality (1.13) by $(R - r)$ and then making $R \to r$, we get the following generalization of inequality (1.9).

**Corollary 1.2** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq k$ and $|z| = 1$, we have

$$\left| zp'(rz) + \frac{n\beta}{r+k} p(rz) \right| \leq \frac{n}{2} \left\{ \left[ k^{-n} \left| r^{n-1} + \frac{\beta}{r+k} r^n \right| + \left| \frac{\beta}{r+k} \right| \right] \max_{|z|=1} \left| p(z) \right| - \left[ k^{-n} \left| r^{n-1} + \frac{\beta}{r+k} r^n \right| - \left| \frac{\beta}{r+k} \right| \right] \min_{|z|=k} \left| p(z) \right| \right\}. \tag{1.14}$$

Taking $\alpha = 0$ in Theorem 1, we also obtain the following generalization of inequality (1.10).

**Corollary 1.3** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$, we have

$$\begin{aligned} &\left| p(Rz) + \beta \left( \frac{R+k}{r+k} \right)^n p(rz) \right| \\ &\quad \leq \frac{1}{2} \left\{ \left[ k^{-n} \left| R^n + \beta \left( \frac{R+k}{r+k} \right)^n r^n \right| + \left| 1 + \beta \left( \frac{R+k}{r+k} \right)^n \right| \right] \max_{|z|=1} \left| p(z) \right| \right. \\ &\quad \left. - \left[ k^{-n} \left| R^n + \beta \left( \frac{R+k}{r+k} \right)^n r^n \right| - \left| 1 + \beta \left( \frac{R+k}{r+k} \right)^n \right| \right] \min_{|z|=k} \left| p(z) \right| \right\}. \end{aligned} \tag{1.15}$$

If we take $\beta = 0$ in Theorem 1, then we have the following consequence.

**Corollary 1.4** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k$ and $|z| = 1$, we have

$$\begin{aligned} \left| p(Rz) - \alpha p(rz) \right| \leq \frac{1}{2} \Big[ & \left\{ k^{-n} \left| R^n - \alpha r^n \right| + |1 - \alpha| \right\} \max_{|z|=1} \left| p(z) \right| \\ & - \left\{ k^{-n} \left| R^n - \alpha r^n \right| - |1 - \alpha| \right\} \min_{|z|=k} \left| p(z) \right| \Big]. \end{aligned} \tag{1.16}$$

If we take $\alpha = 0$ in Corollary 1.4, then we get

**Corollary 1.5** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=R>1} \left| p(z) \right| \leq \left( \frac{R^n + k^n}{2k^n} \right) \max_{|z|=1} \left| p(z) \right| - \left( \frac{R^n - k^n}{2k^n} \right) \min_{|z|=k} \left| p(z) \right|. \tag{1.17}$$

Taking $k = 1$ in Corollary 1.5, inequality (1.17) reduces to inequality (1.8).

## 2 Lemmas

For the proof of Theorem 1, we need the following lemmas. The first lemma is due to Aziz and Zargar [11].

**Lemma 2.1** *If $p(z)$ is a polynomial of degree $n$ having all zeros in the closed disk $|z| \leq k$, where $k \geq 0$, then for every $R \geq r$ and $rR \geq k^2$,*

$$\left|p(Rz)\right| \geq \left(\frac{R+k}{r+k}\right)^n \left|p(rz)\right|, \quad |z| = 1. \tag{2.1}$$

**Lemma 2.2** *Let $F(z)$ be a polynomial of degree $n$ having all zeros in $|z| \leq k$, where $k \geq 0$, and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \leq |F(z)|$ for $|z| = k$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$, we have*

$$\begin{aligned} &\left|f(Rz) - \alpha f(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}f(rz)\right| \\ &\quad \leq \left|F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz)\right|. \end{aligned} \tag{2.2}$$

*Proof* By the inequality $|f(z)| \leq |F(z)|$ for $|z| = k$, any zero of $F(z)$ that lies on $|z| = k$, is a zero of $f(z)$. On the other hand, from Rouche's theorem, it is obvious that for $\delta$ with $|\delta| < 1$, $F(z) + \delta f(z)$ has as many zeros in $|z| < k$ as $F(z)$ does. So, all the zeros of $H(z) := F(z) + \delta f(z)$ lie in $|z| \leq k$. Applying Lemma 2.1, for $R > r \geq k$, we get

$$\left|H(Rz)\right| \geq \left(\frac{R+k}{r+k}\right)^n \left|H(rz)\right| > \left|H(rz)\right|, \quad |z| = 1. \tag{2.3}$$

Therefore, for any $\alpha$ with $|\alpha| \leq 1$, we have

$$\begin{aligned} \left|H(Rz) - \alpha H(rz)\right| &\geq \left|H(Rz)\right| - |\alpha|\left|H(rz)\right| \\ &\geq \left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\left|H(rz)\right|, \quad |z| = 1, \end{aligned}$$

*i.e.*,

$$\left|H(Rz) - \alpha H(rz)\right| \geq \left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\left|H(rz)\right|, \quad |z| = 1. \tag{2.4}$$

Since $H(Rz)$ has all its zeros in $|z| \leq \frac{k}{R} < 1$, a direct application of Rouche's theorem on inequality (2.3) shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Therefore, similar to the first paragraph, it follows that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r \geq k$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}H(rz)$$

lie in $|z| < 1$.

Replacing $H(z)$ by $F(z) + \delta f(z)$, we conclude that all the zeros of

$$\begin{aligned} T(z) &= F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz) \\ &\quad + \delta\left\{f(Rz) - \alpha f(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}f(rz)\right\} \end{aligned} \tag{2.5}$$

lie in $|z| < 1$ for every $R > r \geq k$, $|\alpha| \leq 1$, $|\beta| < 1$ and $|\delta| < 1$. This implies

$$\begin{aligned} &\left|f(Rz) - \alpha f(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}f(rz)\right| \\ &\quad \leq \left|F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz)\right|, \end{aligned} \tag{2.6}$$

where $|z| \geq 1$ and $R > r \geq k$.

If inequality (2.6) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} &\left|f(Rz_0) - \alpha f(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}f(rz_0)\right| \\ &\quad > \left|F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}F(rz_0)\right|. \end{aligned}$$

Take

$$\delta = -\frac{F(Rz_0) - \alpha F(rz_0) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}F(rz_0)}{f(Rz_0) - \alpha f(rz_0) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}f(rz_0)},$$

then $|\delta| < 1$, and with this choice of $\delta$, we have $T(z_0) = 0$ for $|z_0| \geq 1$ from (2.5). But this contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. For $\beta$ with $|\beta| = 1$, (2.6) follows by continuity. This completes the proof. □

If we take $f(z) = (\frac{z}{k})^n \min_{|z|=k} |p(z)|$ in Lemma 2.2, we have

**Lemma 2.3** *If $p(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq k$, where $k \geq 0$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$, we have*

$$\begin{aligned} &k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\min_{|z|=k}|p(z)| \\ &\quad \leq \min_{|z|=1}\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|. \end{aligned} \tag{2.7}$$

If we take $F(z) = (\frac{z}{k})^n \max_{|z|=k} |p(z)|$ in Lemma 2.2, we get

**Lemma 2.4** *Let $p(z)$ be a polynomial of degree at most $n$. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$, we have*

$$\begin{aligned} &\max_{|z|=1}\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\ &\quad \leq k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\max_{|z|=k}|p(z)|. \end{aligned} \tag{2.8}$$

**Lemma 2.5** *If $p(z)$ is a polynomial of degree at most $n$, having no zeros in $|z| < k$, where $k \geq 0$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$, we have*

$$\begin{aligned} &\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\ &\quad \leq \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|, \end{aligned} \tag{2.9}$$

*where $Q(z) = (\frac{z}{k})^n \overline{p(\frac{k^2}{\overline{z}})}$.*

*Proof* If we take $F(z) = Q(z)$ in Lemma 2.2, the result follows. □

**Lemma 2.6** *Let $p(z)$ be a polynomial of degree at most $n$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ $(k \leq 1)$ and $|z| = 1$, we have*

$$\begin{aligned} &\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\ &\qquad + \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right| \\ &\quad \leq \left\{k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\right. \\ &\qquad \left. + \left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right\} \max_{|z|=1}\left|p(z)\right|, \end{aligned} \tag{2.10}$$

*where $Q(z) = (\frac{z}{k})^n \overline{p(k^2/\overline{z})}$.*

*Proof* Let $M = \max_{|z|=k} |p(z)|$. For any $\lambda$ with $|\lambda| > 1$, it follows, by Rouche's theorem, that the polynomial $G(z) = p(z) - \lambda M$ has no zeros in $|z| < k$. Consequently, the polynomial

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\bigl(k^2/\overline{z}\bigr)}$$

has all zeros in $|z| \leq k$ and $|G(z)| = |H(z)|$ for $|z| = k$. On applying Lemma 2.2, for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$, we have that

$$\begin{aligned} &\left|G(Rz) - \alpha G(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}G(rz)\right| \\ &\quad \leq \left|H(Rz) - \alpha H(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}H(rz)\right|. \end{aligned}$$

Therefore, by the equalities

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\overline{z}}\right)} = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\overline{z}}\right)} - \overline{\lambda}\left(\frac{z}{k}\right)^n M = Q(z) - \overline{\lambda}\left(\frac{z}{k}\right)^n M,$$

or

$$H(z) = Q(z) - \overline{\lambda}\left(\frac{z}{k}\right)^n M,$$

and substituting for $G(z)$ and $H(z)$, we get

$$\begin{aligned}
&\left|\left\{p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right\}\right. \\
&\qquad \left. - \lambda\left\{1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right\}M\right| \\
&\quad \leq \left|\left\{Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right\}\right. \\
&\qquad \left. - \overline{\lambda}\left\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right\}k^{-n}Mz^n\right|.
\end{aligned} \tag{2.11}$$

As $|p(z)| = |Q(z)|$ for $|z| = k$, *i.e.*, $M = \max_{|z|=k} |p(z)| = \max_{|z|=k} |Q(z)|$, by Lemma 2.4 for the polynomial $Q(z)$, we obtain

$$\begin{aligned}
&\left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right| \\
&\quad \leq |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}M.
\end{aligned}$$

Therefore, by a suitable choice of argument $\lambda$, we get

$$\begin{aligned}
&\left|\left\{Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right\}\right. \\
&\qquad \left. - \overline{\lambda}\left\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right\}k^{-n}M\right| \\
&\quad = |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}M \\
&\qquad - \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|.
\end{aligned} \tag{2.12}$$

Rewriting the right-hand side of (2.11) by using (2.12), we can obtain

$$\begin{aligned}
&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\
&\qquad - |\lambda|\left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|M \\
&\quad \leq |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}M \\
&\qquad - \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right| \\
&\qquad + \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|
\end{aligned}$$

$$\begin{aligned}&\le |\lambda|\left\{\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}\right.\\&\quad\left.+\left|1-\alpha+\beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right\}M.\end{aligned}$$

Making $|\lambda| \to 1$, we have

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\quad+\left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\\&\le\left\{\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}\right.\\&\quad\left.+\left|1-\alpha+\beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right\}\max_{|z|=k}\left|p(z)\right|.\end{aligned}\tag{2.13}$$

Then by making use of the maximum modulus principle for the polynomial $p(z)$ when $k \le 1$, we get

$$M = \max_{|z|=k}\left|p(z)\right| \le \max_{|z|=1}\left|p(z)\right|.$$

This, in conjunction with (2.13), gives the result. $\square$

**Lemma 2.7** *Let $p(z)$ be a polynomial of degree at most $n$ having no zeros in $|z| < k$, $k \le 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $R > r \ge k$ and $|z| = 1$, we have*

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\le\frac{1}{2}\left\{k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\right.\\&\quad\left.+\left|1-\alpha+\beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right\}\max_{|z|=1}\left|p(z)\right|.\end{aligned}\tag{2.14}$$

*Proof* Since $p(z)$ does not vanish in $|z| < k$, applying Lemma 2.5 yields

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\le\left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|,\end{aligned}\tag{2.15}$$

where $Q(z) = (\frac{z}{k})^n\overline{p(k^2/\overline{z})}$.

Combining inequalities (2.15) and (2.10), we have

$$\begin{aligned}&2\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\le\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\end{aligned}$$

$$+ \left| Q(Rz) - \alpha Q(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} Q(rz) \right|$$
$$\leq \left\{ \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n k^{-n} z^n \right| \right.$$
$$\left. + \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} \right| \right\} \max_{|z|=1} |p(z)|. \tag{2.16}$$

This gives the result. □

## 3 Proof of the theorem

The proof follows some known ideas in the literature.

*Proof of Theorem* 1 If $p(z)$ has a zero on $|z| = k$, then the result follows from Lemma 2.7. Therefore, we assume that $p(z)$ has all zeros in $|z| > k$. Then $m = \min_{|z|=k} |p(z)| > 0$ and for $\lambda$ with $|\lambda| < 1$, we have $|\lambda m| < m \leq |p(z)|$, where $|z| = k$. Using Rouche's theorem, we conclude that the polynomial $G(z) = p(z) - \lambda m$ has no zeros in $|z| < k$. Consequently, the polynomial

$$H(z) = \left( \frac{z}{k} \right)^n \overline{G(k^2/\overline{z})}$$

has all its zeros in $|z| \leq k$ and $|G(z)| = |H(z)|$ for $|z| = k$. Therefore, by applying Lemma 2.2 for the polynomials $G(z)$ and $H(z)$, we obtain

$$\left| G(Rz) - \alpha G(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} G(rz) \right|$$
$$\leq \left| H(Rz) - \alpha H(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} H(rz) \right|. \tag{3.1}$$

Using the fact that

$$H(z) = \left( \frac{z}{k} \right)^n \overline{G\left( \frac{k^2}{\overline{z}} \right)} = \left( \frac{z}{k} \right)^n \overline{p\left( \frac{k^2}{\overline{z}} \right)} - \overline{\lambda} m \left( \frac{z}{k} \right)^n = Q(z) - \overline{\lambda} m \left( \frac{z}{k} \right)^n,$$

or

$$H(z) = Q(z) - \overline{\lambda} m \left( \frac{z}{k} \right)^n,$$

and substituting for $G(z)$ and $H(z)$ in (3.1), we get

$$\left| \left\{ p(Rz) - \alpha p(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} p(rz) \right\} - \lambda \left\{ 1 - \alpha + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} \right\} m \right|$$
$$\leq \left| \left\{ Q(Rz) - \alpha Q(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} Q(rz) \right\} \right.$$
$$\left. - \overline{\lambda} \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n \right\} k^{-n} m z^n \right|. \tag{3.2}$$

Since the polynomial $Q(z) = (\frac{z}{k})^n\overline{p(\frac{k^2}{\bar{z}})}$ has all zeros in $|z| \leq k$ and $m = \min_{|z|=k}|p(z)| = \min_{|z|=k}|Q(z)|$, by applying Lemma 2.3 for the polynomial $Q(z)$, one can obtain

$$\begin{aligned}&\left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\\&\quad \geq |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}m.\end{aligned}$$

Therefore, by a suitable choice of argument $\lambda$, we get

$$\begin{aligned}&\left|\left\{Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right\}\right.\\&\qquad \left.- \bar{\lambda}\left\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right\}\right|k^{-n}m\\&\quad = \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\\&\qquad - |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}m. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have that

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\qquad - |\lambda|\left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|m\\&\quad \leq \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\\&\qquad - |\lambda|\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|k^{-n}m.\end{aligned}$$

This implies

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\quad \leq \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\\&\qquad - |\lambda|\left\{k^{-n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\right|\right.\\&\qquad \left. - \left|1 - \alpha + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}\right|\right\}m.\end{aligned}$$

Making $|\lambda| \to 1$, one obtains

$$\begin{aligned}&\left|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)\right|\\&\quad \leq \left|Q(Rz) - \alpha Q(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}Q(rz)\right|\end{aligned}$$

$$
\begin{aligned}
&- \left\{ k^{-n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n \right| \right. \\
&\left. - \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} \right| \right\} m. \qquad (3.4)
\end{aligned}
$$

On the other hand, by Lemma 2.6, we have

$$
\begin{aligned}
&\left| p(Rz) - \alpha p(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} p(rz) \right| \\
&\quad + \left| Q(Rz) - \alpha Q(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} Q(rz) \right| \\
&\le \left\{ k^{-n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n \right| \right. \\
&\quad \left. - \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} \right| \right\} \max_{|z|=1} \left| p(z) \right|. \qquad (3.5)
\end{aligned}
$$

Considering inequalities (3.4) and (3.5) together gives the result. □

**Competing interests**
The author declares that they have no competing interests.

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